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2005 J. Phys. A: Math. Gen. 38 L405

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## LETTER TO THE EDITOR

## Fifth-neighbour spin–spin correlator for the anti-ferromagnetic Heisenberg chain

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Received 1 April 2005

Published 10 May 2005

Online at [stacks.iop.org/JPhysA/38/L405](http://stacks.iop.org/JPhysA/38/L405)

### Abstract

We study the generating function of the spin–spin correlation functions in the ground state of the anti-ferromagnetic spin-1/2 Heisenberg chain without magnetic field. We have found its fundamental functional relations from those for general correlation functions, which originate in the quantum Knizhnik–Zamolodchikov equation. Using these relations, we have calculated the explicit form of the generating functions up to  $n = 6$ . Accordingly we can obtain the spin–spin correlator  $\langle S_j^z S_{j+k}^z \rangle$  up to  $k = 5$ .

PACS numbers: 75.10.Jm, 02.30.Ik, 75.50.Ee

Recently there have been rapid developments in the investigation of the exact calculation of the correlation functions for the spin-1/2 Heisenberg chains. In particular for the ground state without magnetic field, explicit analytical forms of several correlation functions have been calculated in the thermodynamic limit. In this letter we shall report further results for exact calculation of correlation functions for the anti-ferromagnetic Heisenberg  $XXX$  chain, whose Hamiltonian is given by

$$\mathcal{H} = \sum_{j=-\infty}^{\infty} [S_j^x S_{j+1}^x + S_j^y S_{j+1}^y + S_j^z S_{j+1}^z], \quad (1)$$

where  $S_j^\alpha = \sigma_j^\alpha / 2$ , with  $\sigma_j^\alpha$  being the Pauli matrices acting on the  $j$ th site. The Hamiltonian (1) can be diagonalized by the Bethe ansatz, and many bulk physical quantities have been evaluated in the thermodynamic limit [1, 2]. On the other hand, the exact calculation of the correlation functions is still an ongoing problem.

First, let us list the known exact results of the two-point spin–spin correlators  $\langle S_j^z S_{j+k}^z \rangle$ , which are physically the most important correlation functions:

$$\langle S_j^z S_{j+1}^z \rangle = \frac{1}{12} - \frac{1}{3} \zeta_a(1) = -0.147715726853315\dots, \quad (2)$$

$$\langle S_j^z S_{j+2}^z \rangle = \frac{1}{12} - \frac{4}{3} \zeta_a(1) + \zeta_a(3) = 0.060\,679\,769\,956\,435 \dots, \quad (3)$$

$$\begin{aligned} \langle S_j^z S_{j+3}^z \rangle &= \frac{1}{12} - 3\zeta_a(1) + \frac{74}{9} \zeta_a(3) - \frac{56}{9} \zeta_a(1)\zeta_a(3) - \frac{8}{3} \zeta_a(3)^2 - \frac{50}{9} \zeta_a(5) + \frac{80}{9} \zeta_a(1)\zeta_a(5) \\ &= -0.050\,248\,627\,257\,235 \dots, \end{aligned} \quad (4)$$

$$\begin{aligned} \langle S_j^z S_{j+4}^z \rangle &= \frac{1}{12} - \frac{16}{3} \zeta_a(1) + \frac{290}{9} \zeta_a(3) - 72\zeta_a(1)\zeta_a(3) - \frac{1172}{9} \zeta_a(3)^2 - \frac{700}{9} \zeta_a(5) \\ &\quad + \frac{4640}{9} \zeta_a(1)\zeta_a(5) - \frac{220}{9} \zeta_a(3)\zeta_a(5) - \frac{400}{3} \zeta_a(5)^2 \\ &\quad + \frac{455}{9} \zeta_a(7) - \frac{3920}{9} \zeta_a(1)\zeta_a(7) + 280\zeta_a(3)\zeta_a(7) \\ &= 0.034\,652\,776\,982\,728 \dots \end{aligned} \quad (5)$$

Here  $\zeta_a(s)$  is the alternating zeta function defined by  $\zeta_a(s) \equiv \sum_{n=1}^{\infty} (-1)^{n-1}/n^s$ , which is related to the Riemann zeta function  $\zeta(s) = \zeta_a(s)/(1 - 2^{1-s})$ . Note that the alternating zeta function is regular at  $s = 1$  and is given by  $\zeta_a(1) = \ln 2$ . Thus the analytical form of the spin–spin correlators  $\langle S_j^z S_{j+k}^z \rangle$  has been obtained up to  $k = 4$  so far. The nearest-neighbour correlator (2) was obtained from Hulthén’s result for the ground state energy in 1938 [3]. The second-neighbour correlator (3) was obtained by Takahashi in 1977 [4] via the strong coupling expansion of the ground state energy for the half-filled Hubbard chain. After that it had taken a long time before the explicit form of the third-neighbour correlator (4) was obtained by Sakai, Shiroishi, Nishiyama and Takahashi in 2003 [5]. They applied the Boos–Korepin method [6, 7] to evaluate the multiple integral formula for correlation functions developed by the Kyoto group [8–12] (see also [13]). The Boos–Korepin method was originally devised to calculate a special correlation function called the emptiness formation probability (EFP) [10], which is defined by

$$P(n) \equiv \left\langle \prod_{j=1}^n \left( \frac{1}{2} + S_j^z \right) \right\rangle. \quad (6)$$

This is the probability of finding a ferromagnetic string of length  $n$  in the ground state. The multiple integral formula for  $P(n)$  as well as other correlation functions among the adjacent  $n$ -spins consists of  $n$ -dimensional integrals.

By use of the Boos–Korepin method, we can reduce the multiple integrals to one-dimensional ones by transforming the integrand to a certain canonical form. In the case of the present  $XXX$  model, the remaining one-dimensional integrals can be further expressed in terms of  $\zeta_a(s)$ ,  $s \in \mathbb{Z}_{\geq 1}$ . Hence, *in principle*, we can obtain the explicit form of arbitrary correlation functions by the Boos–Korepin method. However, *practically* it becomes a tremendously hard task to calculate the canonical form as the integral dimension increases. Actually  $P(5)$  is the only correlation function which was calculated by the Boos–Korepin method among those for five lattice sites [14].

To overcome such a difficulty, Boos, Korepin and Smirnov invented an alternative method to obtain  $P(n)$  [15]. They considered the inhomogeneous  $XXX$  model, where each site carries an inhomogeneous parameter  $\lambda_j$ . Accordingly the correlation functions depend on the parameters  $\lambda_j$ , for example,  $P_n(\lambda_1, \dots, \lambda_n)$ . Boos, Korepin and Smirnov derived simple functional relations, which  $P_n(\lambda_1, \dots, \lambda_n)$  should satisfy from the underlying quantum Knizhnik–Zamolodchikov (qKZ) equations. Together with a simple ansatz for the final form of the  $P_n(\lambda_1, \dots, \lambda_n)$ , they have shown the problem reduces to solving large linear systems

of equations for rational coefficients. In this way they could calculate  $P_n(\lambda_1, \dots, \lambda_n)$  up to  $n = 6$ . In particular, by taking the homogeneous limit  $\lambda_j \rightarrow 0$ , they obtained the analytical form of  $P(n)$  up to  $n = 6$ . Thus this new method is more powerful than the original Boos–Korepin method to evaluate the multiple integrals. It was recently generalized to calculate arbitrary inhomogeneous correlation functions by Boos, Shiroishi and Takahashi [16]. They calculated all the independent correlation functions among five lattice sites and especially the fourth-neighbour correlator (5). In this letter we explore these functional methods further. In particular, the main new aspect is to introduce the generating function of the spin–spin correlators [17–21] into the scheme, which allows us to calculate the  $\langle S_j^z S_{j+k}^z \rangle$  more efficiently.

We remark that Boos, Jimbo, Miwa, Smirnov and Takeyama have obtained more explicit recursion relations for arbitrary correlation functions [22] (see also [23]) by investigating the qKZ equations more profoundly. Their formulae have proven the ansatz for the final form of the inhomogeneous correlation functions. However, the formulae contain unusual trace functions with a *fractional* dimension, the effective evaluation of which has not yet been developed.

Here we define the generating function by

$$P_n^\kappa(\lambda_1, \lambda_2, \dots, \lambda_n) \equiv \left\langle \prod_{j=1}^n \left\{ \left( \frac{1}{2} + S_j^z \right) + \kappa \left( \frac{1}{2} - S_j^z \right) \right\} \right\rangle (\lambda_1, \lambda_2, \dots, \lambda_n), \tag{7}$$

where  $\kappa$  is a parameter. Once we obtain the generating functions, we can calculate the two-point spin–spin correlators through the relation [20]

$$\begin{aligned} \langle S_1^z S_{k+1}^z \rangle (\lambda_1, \dots, \lambda_{k+1}) &= \frac{1}{2} \frac{\partial^2}{\partial \kappa^2} \left\{ P_{k+1}^\kappa(\lambda_1, \lambda_2, \dots, \lambda_{k+1}) - P_k^\kappa(\lambda_1, \dots, \lambda_k) \right. \\ &\quad \left. - P_k^\kappa(\lambda_2, \dots, \lambda_{k+1}) + P_{k-1}^\kappa(\lambda_2, \dots, \lambda_k) \right\} \Big|_{\kappa=1} - \frac{1}{4}, \\ \langle S_1^z S_{k+1}^z \rangle &= \lim_{\lambda_j \rightarrow 0} \langle S_1^z S_{k+1}^z \rangle (\lambda_1, \dots, \lambda_{k+1}). \end{aligned} \tag{8}$$

Note that  $P_n^\kappa(\lambda_1, \lambda_2, \dots, \lambda_n)$  is a natural generalization of the EFP as

$$P_n^{\kappa=0}(\lambda_1, \lambda_2, \dots, \lambda_n) = P_n(\lambda_1, \lambda_2, \dots, \lambda_n), \tag{9}$$

with other relations

$$P_n^{\kappa=1}(\lambda_1, \lambda_2, \dots, \lambda_n) = 1, \tag{10}$$

$$P_n^{\kappa=-1}(\lambda_1, \lambda_2, \dots, \lambda_n) = 2^n \left\langle \prod_{j=1}^n S_j^z \right\rangle (\lambda_1, \lambda_2, \dots, \lambda_n). \tag{11}$$

From the qKZ equations for the correlation functions [16, 22, 23], we find that the generating function satisfies the following functional relations.

- Translational invariance

$$P_n^\kappa(\lambda_1 + x, \dots, \lambda_n + x) = P_n^\kappa(\lambda_1, \dots, \lambda_n). \tag{12}$$

- Negating relation

$$P_n^\kappa(\lambda_1, \dots, \lambda_n) = P_n^\kappa(-\lambda_1, \dots, -\lambda_n). \tag{13}$$

- Symmetry relation

$$P_n^\kappa(\lambda_1, \dots, \lambda_j, \lambda_{j+1}, \dots, \lambda_n) = P_n^\kappa(\lambda_1, \dots, \lambda_{j+1}, \lambda_j, \dots, \lambda_n). \tag{14}$$

- First recurrent relation

$$P_n^\kappa(\lambda_1, \dots, \lambda_{n-1}, \lambda_{n-1} \pm 1) = \kappa P_{n-2}^\kappa(\lambda_1, \dots, \lambda_{n-2}). \quad (15)$$

- Second recurrent relation

$$\lim_{\lambda_n \rightarrow i\infty} P_n^\kappa(\lambda_1, \dots, \lambda_{n-1}, \lambda_n) = \frac{1+\kappa}{2} P_{n-1}^\kappa(\lambda_1, \dots, \lambda_{n-1}). \quad (16)$$

Here and hereafter we follow the notations in [22] for the inhomogeneous parameter  $\lambda_j$  and also the transcendental function  $\omega(\lambda)$  below. One easily sees that these functional relations reduce to those for the EFP,  $P_n(\lambda_1, \dots, \lambda_n)$  [15], if we set  $\kappa = 0$ . According to the (proven) general form of the correlation functions and also by the symmetry relation (14), we can assume the generating function in the form

$$P_n^\kappa(\lambda_1, \dots, \lambda_n) = \sum_{l=0}^{\lfloor \frac{n}{2} \rfloor} \left\{ A_{n,l}^\kappa(\lambda_1, \lambda_2 | \dots | \lambda_{2l-1}, \lambda_{2l} | \lambda_{2l+1}, \dots, \lambda_n) \right. \\ \left. \times \prod_{j=1}^l \omega(\lambda_{2j-1} - \lambda_{2j}) + \text{permutations} \right\}, \quad (17)$$

where

$$\omega(\lambda) = \sum_{k=1}^{\infty} (-1)^k \frac{2k(\lambda^2 - 1)}{\lambda^2 - k^2} + \frac{1}{2} \quad (18)$$

is a generating function of the alternating zeta values

$$\omega(\lambda) = 2(\lambda^2 - 1) \sum_{k=0}^{\infty} \lambda^{2k} \zeta_a(2k+1) + \frac{1}{2}. \quad (19)$$

Moreover the function  $\omega(\lambda)$  has the following properties:

$$\omega(i\infty) = 0, \quad \omega(\lambda \pm 1) = \alpha(\lambda) + \gamma_{\pm}(\lambda)\omega(\lambda), \quad (20)$$

where

$$\alpha(\lambda) = \frac{3}{2} \frac{1}{\lambda^2 - 1}, \quad \gamma_{\pm}(\lambda) = -\frac{\lambda(\lambda \pm 2)}{\lambda^2 - 1}. \quad (21)$$

Note also that  $A_{n,l}^\kappa(\lambda_1, \lambda_2 | \dots | \lambda_{2l-1}, \lambda_{2l} | \lambda_{2l+1}, \dots, \lambda_n)$  are rational functions depending on the parameter  $\kappa$  with known denominators

$$A_{n,l}^\kappa(\lambda_1, \lambda_2 | \dots | \lambda_{2l-1}, \lambda_{2l} | \lambda_{2l+1}, \dots, \lambda_n) = \frac{\prod_{k=1}^l \lambda_{2k-1, 2k} \prod_{2l+1 \leq k < j \leq n} \lambda_{kj}}{\prod_{1 \leq k < j \leq n} \lambda_{kj}} \\ \times Q_{n,l}^\kappa(\lambda_1, \lambda_2 | \dots | \lambda_{2l-1}, \lambda_{2l} | \lambda_{2l+1}, \dots, \lambda_n), \quad (22)$$

where  $\lambda_{kj} = \lambda_k - \lambda_j$ , and  $Q_{n,l}^\kappa(\lambda_1, \lambda_2 | \dots | \lambda_{2l-1}, \lambda_{2l} | \lambda_{2l+1}, \dots, \lambda_n)$  are some polynomials.

Our proposal is to determine  $A_{n,l}^\kappa(\lambda_1, \lambda_2 | \dots | \lambda_{2l-1}, \lambda_{2l} | \lambda_{2l+1}, \dots, \lambda_n)$  and therefore the explicit form of the generating functions from the functional relations (12)–(16) together with the ansatz (17). Simple calculations yield that the second recurrent relation (16) is equivalent to the following recursion equation for  $A_{n,l}^\kappa$ :

$$\lim_{\lambda_n \rightarrow i\infty} A_{n,l}^\kappa(\lambda_1, \lambda_2 | \dots | \lambda_{2l-1}, \lambda_{2l} | \lambda_{2l+1}, \dots, \lambda_n) \\ = \frac{1+\kappa}{2} A_{n-1,l}^\kappa(\lambda_1, \lambda_2 | \dots | \lambda_{2l-1}, \lambda_{2l} | \lambda_{2l+1}, \dots, \lambda_{n-1}). \quad (23)$$

Note that it is easy to see

$$A_{n,0}^\kappa(|\lambda_1, \dots, \lambda_n) = \left(\frac{1+\kappa}{2}\right)^n \tag{24}$$

for any  $n$ . The first recurrent relation (15) cannot be reduced to a general formula for  $A_{n,l}^\kappa$ . Instead, we write down the equations for  $A_{n,l}^\kappa$  corresponding to the first recurrent relation up to  $n = 3$ :

$$\left(\frac{1+\kappa}{2}\right)^2 - \frac{3}{2}A_{2,1}^\kappa(\lambda_1, \lambda_1 \pm 1|) = \kappa, \tag{25}$$

$$\left(\frac{1+\kappa}{2}\right)^3 + \alpha(\lambda_{12})A_{3,1}^\kappa(\lambda_1, \lambda_2 \pm 1|\lambda_2) - \frac{3}{2}A_{3,1}^\kappa(\lambda_2, \lambda_2 \pm 1|\lambda_1) = \kappa \left(\frac{1+\kappa}{2}\right), \tag{26}$$

$$A_{3,1}^\kappa(\lambda_1, \lambda_2|\lambda_2 \pm 1) + \gamma_{\mp}(\lambda_{12})A_{3,1}^\kappa(\lambda_1, \lambda_2 \pm 1|\lambda_2) = 0. \tag{27}$$

Solving these equations, we have calculated the explicit forms of  $Q_{n,l}^\kappa$  up to  $n = 5$ , which are given by

$$\begin{aligned} Q_{2,1}^\kappa(\lambda_1, \lambda_2|) &= \frac{1}{6}(1-\kappa)^2, & Q_{3,1}^\kappa(\lambda_1, \lambda_2|\lambda_3) &= \frac{1+\kappa}{2}(1+\lambda_{13}\lambda_{23})Q_{2,1}^\kappa(\lambda_1, \lambda_2|), \\ Q_{4,1}^\kappa(\lambda_1, \lambda_2|\lambda_3, \lambda_4) &= \frac{1+\kappa}{2}(1+\lambda_{14}\lambda_{24})Q_{3,1}^\kappa(\lambda_1, \lambda_2|\lambda_3) - \frac{(1-\kappa)^4}{120}(\lambda_{12}^2 - 4), \\ Q_{4,2}^\kappa(\lambda_1, \lambda_2|\lambda_3, \lambda_4|) &= \frac{1}{24}\left(1+\frac{\kappa}{3}\right)(1+3\kappa)(1-\kappa)^2 + \frac{(1-\kappa)^4}{90}\left(\lambda_{12}^2 - \frac{3}{2}\right)\left(\lambda_{34}^2 - \frac{3}{2}\right) \\ &\quad + \frac{(1+\kappa)^2(1-\kappa)^2}{36}(1+\lambda_{13}\lambda_{24})(1+\lambda_{14}\lambda_{23}) \\ &\quad - \frac{\kappa(1-\kappa)^2}{18}(1-\lambda_{13}\lambda_{24})(1-\lambda_{14}\lambda_{23}), \\ Q_{5,1}^\kappa(\lambda_1, \lambda_2|\lambda_3, \lambda_4, \lambda_5) &= \frac{1+\kappa}{2}(1+\lambda_{15}\lambda_{25})Q_{4,1}^\kappa(\lambda_1, \lambda_2|\lambda_3, \lambda_4) \\ &\quad - \frac{1+\kappa}{2}\frac{(1-\kappa)^4}{120}(\lambda_{12}^2 - 4)(4+\lambda_{13}\lambda_{23}+\lambda_{14}\lambda_{24}), \\ Q_{5,2}^\kappa(\lambda_1, \lambda_2|\lambda_3, \lambda_4|\lambda_5) &= \frac{1+\kappa}{2}(1+\lambda_{15}\lambda_{25})(1+\lambda_{35}\lambda_{45})Q_{4,2}^\kappa(\lambda_1, \lambda_2|\lambda_3, \lambda_4|) \\ &\quad + \frac{(1-\kappa)^2}{360}(\lambda_{12}^2 - 4)(\lambda_{34}^2 - 4)\{(1+\kappa)(1-\kappa)^2(\lambda_{14}\lambda_{23}+\lambda_{13}\lambda_{24}) \\ &\quad + 5(1+\kappa+\kappa^2+\kappa^3)\} - \frac{\kappa(1+\kappa)(1-\kappa)^2}{36}\{10 - 2(\lambda_{12}^2 + \lambda_{34}^2 + \lambda_{14}\lambda_{23}\lambda_{13}\lambda_{24}) \\ &\quad + (\lambda_{14}\lambda_{23} + \lambda_{13}\lambda_{24} + 2)(\lambda_{15}\lambda_{25} + \lambda_{35}\lambda_{45} - 1)\}. \end{aligned} \tag{28}$$

One can confirm that the previous known results of two-point spin–spin correlators (2), . . . , (5) are reproduced through the relation (8). In a similar way, we have calculated  $P_6^\kappa(\lambda_1, \dots, \lambda_6)$ . Since its expression is too complicated, we present here only the final two new results of correlation functions among six lattice sites. The first one is the two-point fifth-neighbour spin–spin correlator,

$$\begin{aligned} \langle S_j^z S_{j+5}^z \rangle &= \frac{1}{12} - \frac{25}{3}\zeta_a(1) + \frac{800}{9}\zeta_a(3) - \frac{1192}{3}\zeta_a(1)\zeta_a(3) - \frac{15\,368}{9}\zeta_a(3)^2 - 608\zeta_a(3)^3 \\ &\quad - \frac{4228}{9}\zeta_a(5) + \frac{64\,256}{9}\zeta_a(1)\zeta_a(5) - \frac{976}{9}\zeta_a(3)\zeta_a(5) + 3648\zeta_a(1)\zeta_a(3)\zeta_a(5) \end{aligned}$$

$$\begin{aligned}
& -\frac{3328}{3}\zeta_a(3)^2\zeta_a(5) - \frac{76640}{3}\zeta_a(5)^2 + \frac{66560}{3}\zeta_a(1)\zeta_a(5)^2 \\
& + \frac{12640}{3}\zeta_a(3)\zeta_a(5)^2 + \frac{6400}{3}\zeta_a(5)^3 + \frac{9674}{9}\zeta_a(7) - \frac{225848}{9}\zeta_a(1)\zeta_a(7) \\
& + 56952\zeta_a(3)\zeta_a(7) - \frac{116480}{3}\zeta_a(1)\zeta_a(3)\zeta_a(7) - \frac{35392}{3}\zeta_a(3)^2\zeta_a(7) \\
& + 7840\zeta_a(5)\zeta_a(7) - 8960\zeta_a(3)\zeta_a(5)\zeta_a(7) - \frac{66640}{3}\zeta_a(7)^2 \\
& + 31360\zeta_a(1)\zeta_a(7)^2 - 686\zeta_a(9) + 18368\zeta_a(1)\zeta_a(9) - 53312\zeta_a(3)\zeta_a(9) \\
& + 35392\zeta_a(1)\zeta_a(3)\zeta_a(9) + 16128\zeta_a(3)^2\zeta_a(9) \\
& + 38080\zeta_a(5)\zeta_a(9) - 53760\zeta_a(1)\zeta_a(5)\zeta_a(9) \\
& = -0.030890366647609\dots
\end{aligned} \tag{29}$$

Another one is a six-spin correlation function

$$\begin{aligned}
\lim_{\lambda_j \rightarrow 0} P_6^{\kappa=-1}(\lambda_1, \dots, \lambda_6) &= 2^6 \left\langle \prod_{j=1}^6 S_j^z \right\rangle \\
&= \frac{1}{7} - 12\zeta_a(1) + \frac{1112}{5}\zeta_a(3) - \frac{3776}{3}\zeta_a(1)\zeta_a(3) - \frac{100736}{15}\zeta_a(3)^2 \\
& - \frac{352768}{135}\zeta_a(3)^3 - \frac{71656}{35}\zeta_a(5) + \frac{442496}{15}\zeta_a(1)\zeta_a(5) + \frac{15104}{15}\zeta_a(3)\zeta_a(5) \\
& + \frac{705536}{45}\zeta_a(1)\zeta_a(3)\zeta_a(5) - \frac{212992}{45}\zeta_a(3)^2\zeta_a(5) - \frac{6796736}{63}\zeta_a(5)^2 \\
& + \frac{851968}{9}\zeta_a(1)\zeta_a(5)^2 + \frac{161792}{9}\zeta_a(3)\zeta_a(5)^2 + \frac{1723520}{189}\zeta_a(5)^3 + \frac{32432}{5}\zeta_a(7) \\
& - \frac{350336}{3}\zeta_a(1)\zeta_a(7) + 241888\zeta_a(3)\zeta_a(7) - \frac{1490944}{9}\zeta_a(1)\zeta_a(3)\zeta_a(7) \\
& - \frac{2265088}{45}\zeta_a(3)^2\zeta_a(7) + \frac{312064}{9}\zeta_a(5)\zeta_a(7) - \frac{344704}{9}\zeta_a(3)\zeta_a(5)\zeta_a(7) \\
& - \frac{833168}{9}\zeta_a(7)^2 + \frac{1206464}{9}\zeta_a(1)\zeta_a(7)^2 - \frac{23256}{5}\zeta_a(9) + \frac{443008}{5}\zeta_a(1)\zeta_a(9) \\
& - \frac{3437248}{15}\zeta_a(3)\zeta_a(9) + \frac{2265088}{15}\zeta_a(1)\zeta_a(3)\zeta_a(9) + \frac{344704}{5}\zeta_a(3)^2\zeta_a(9) \\
& + \frac{476096}{3}\zeta_a(5)\zeta_a(9) - \frac{689408}{3}\zeta_a(1)\zeta_a(5)\zeta_a(9) \\
& = -0.440301669702626\dots
\end{aligned} \tag{30}$$

One can see that these analytical expressions have the same structure as  $P(6)$  previously obtained in [15]. Namely they are the polynomials of  $\zeta_a(1), \dots, \zeta_a(9)$  of degree 3 with different rational coefficients. We have confirmed these results (29) and (30) by comparing with the numerical data by the exact diagonalization for the finite systems.

In summary we have presented a new effective method to calculate the analytical form of the spin–spin correlators starting from the generating function. In particular, we can obtain

the fifth-neighbour correlator  $\langle S_j^z S_{j+5}^z \rangle$ . The details of the calculations and further results will be reported in a separate publication.

### Acknowledgments

We are grateful to K Sakai, M Takahashi and Z Tsuboi for valuable discussions. This work is supported in part by grant-in-aid for the Scientific Research (B) no 14340099 from the Ministry of Education, Culture, Sports, Science and Technology, Japan. MS is supported by grant-in-aid for young scientists no 14740228.

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